

A note on Blasius type boundary value problems

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September 13, 2011

Abstract

The existence and uniqueness of a solution to a generalized Blasius equation with asymptotic boundary conditions are proved. A new numerical approximation method is proposed.

keywords: Blasius equation, shooting method

2010 MSC: 34B15, 34B40, 34D05, 65L10

1 Introduction

We study the BVP of the form:

$$x''' + cx^p \cdot x'' = 0, \quad x(0) = 0 = x'(0), \quad \lim_{t \rightarrow \infty} x'(t) = \beta, \quad (1.1)$$

where $p \geq 1$, c and β are positive constants. The problem is motivated by the classical Blasius equation describing the velocity profile of the fluid in the boundary layer where $c = \frac{1}{2}$, $p = \beta = 1$. The Blasius equation is a basic equation in fluid mechanics which appears in the study of the flow of an incompressible viscous fluid over semi infinite plane. Blasius ([1]) used a similarity transform technique to convert the partial differential equation into his famous ordinary differential equation

$$x''' + \frac{1}{2}x \cdot x'' = 0, \quad x(0) = 0 = x'(0), \quad \lim_{t \rightarrow \infty} x'(t) = 1, \quad (1.2)$$

where x is the stream function $x = \frac{\Psi}{\sqrt{2U\nu y_1}}$, U is the fluid velocity, ν is the fluid kinematic viscosity and t is the similarity variable defined as $t = y_2 \sqrt{\frac{U}{2\nu y_1}}$, where y_1, y_2 are Cartesian coordinates with y_1 pointing along the free stream direction and y_2 perpendicular to y_1 . We refer to [2, 3] for an excellent introduction to the problem. A series expansions method was used to solve (1.2) by Blasius. There has been appeared many analytical and numerical methods handling this problem since the Blasius's work, [6, 8] for instance.

In the first part of this paper, the existence and uniqueness of (1.1) will be analytically proved by changing the boundary value problem to an initial problem. Using the obtained estimates we will be able to find the value of $a = x''(0)$ which guarantees that the solution x_a on an initial problem

$$x''' + cx^p \cdot x'' = 0, \quad x(0) = 0 = x'(0), \quad x''(0) = a, \quad (1.3)$$

is the solution of (1.1) we are looking for. In the second part of the article, a new numerical approximation method is proposed.

2 Auxiliary lemmas

Let x_a stand for the unique solution satisfying initial conditions

$$x(0) = 0 = x'(0), \quad x''(0) = a.$$

If $a < 0$, then x_a and x_a'' are negative for small t 's thus the solution is concave and negative for all arguments and it cannot solve (1.1). For $a = 0$ we have a trivial solution $x_a \equiv 0$ and the seeking solution can be obtained for $a > 0$.

Lemma 1. *The x_a is defined for all $t \geq 0$.*

Proof. x_a'' cannot vanish at any point t_0 by the uniqueness of solutions of initial value problems: $x(t) = c_1 t + c_2$ solves our ODE. Hence dividing the equation by x_a'' and integrating on $[0, t]$ we have

$$x_a''(t) = a \exp \left(-c \int_0^t x_a(s)^p ds \right), \quad (2.1)$$

which implies

$$x_a'(t) = a \int_0^t \exp \left(-c \int_0^s x_a(\tau)^p d\tau \right) ds. \quad (2.2)$$

Integrating once more and applying the Fubini Theorem we get

$$x_a(t) = a \int_0^t (t-s) \exp \left(-c \int_0^s x_a(\tau)^p d\tau \right) ds. \quad (2.3)$$

By (2.1), (2.2) and (2.3) we have apriori estimates:

$$0 < x_a(t) < \frac{1}{2}at^2, \quad 0 < x_a'(t) < at, \quad 0 < x_a''(t) < a$$

for any $t > 0$. It follows ([7], p. 146) that x_a is extendable to $[0, \infty)$.

Lemma 2. *For any $a > 0$, $\lim_{t \rightarrow \infty} x_a''(t) = 0$ and there exists a finite and positive limit $h(a) := \lim_{t \rightarrow \infty} x_a'(t)$.*

Proof. Since $x_a'' > 0$ and $x_a' > 0$, then $\lim_{t \rightarrow +\infty} x_a(t) = +\infty$. Moreover, since $x_a'' > 0$, $x_a > 0$ and $x_a''' = -cx_a^p x_a''$, then x_a'' is a decreasing function so $\lim_{t \rightarrow \infty} x_a''(t) = g_a \in [0, a)$. Suppose $g_a > 0$. From $x_a''' = -cx_a^p x_a''$ we get that

$$\lim_{t \rightarrow +\infty} x_a'''(t) = -\infty. \quad (2.4)$$

On the other hand

$$\forall t \geq 0 \exists s_t \in (t, t+1) \quad x_a''(t+1) - x_a''(t) = x_a'''(s_t)$$

and hence $\lim_{t \rightarrow +\infty} x_a'''(t) = 0$, which contradicts (2.4).

Since $x_a'' > 0$, then x_a' is an increasing function (so the limit defining $h(a)$ exists, possibly infinite). From $\lim_{t \rightarrow +\infty} x_a(t) = +\infty$ we get there exists $t_a > 0$ such that $cx_a(t)^p > 1$ for $t > t_a$. From (2.2), we obtain for $t > t_a$,

$$\begin{aligned} x_a'(t) &\leq a \int_0^t \exp\left(-c \int_0^\tau x_a(\tau)^p d\tau\right) \exp\left(-\int_{t_a}^s d\tau\right) ds \leq \\ &\leq a \int_0^{t_a} ds + a \int_{t_a}^t e^{-(s-t_a)} ds \leq at_a + ae^{t_a} \int_{t_a}^\infty e^{-s} ds = a(t_a + 1). \end{aligned}$$

Thus

$$h(a) \leq a(t_a + 1) \quad (2.5)$$

for any $a > 0$.

Lemma 3. *For any $a > 0$, there exists a finite and positive limit $\mu(a) := \lim_{t \rightarrow \infty} (h(a)t - x_a(t))$. It means that the graph of x_a has a slant asymptote and the following estimates hold:*

$$\max(0, h(a)t - \mu(a)) \leq x_a(t) \leq h(a)t. \quad (2.6)$$

Proof. The function $t \mapsto h(a)t - x_a(t)$ is increasing, hence the limit from the assertion exists but it can be infinite. Suppose it equals $+\infty$. By the arguments from the proof of the lemma 2, we have $x_a'''(t) \leq -x_a''(t)$ for $t \geq t_a$. Integrating this inequality from s to $+\infty$ and using the fact $x_a''(+\infty) = 0$, we get

$$-x_a''(s) \leq -h(a) + x_a'(s).$$

Next integration from t_a to t leads to the following inequality

$$x_a'(t_a) - x_a'(t) \leq -h(a)(t - t_a) + x_a(t) - x_a(t_a)$$

or equivalently

$$h(a)t - x_a(t) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a).$$

Thus

$$0 < \mu(a) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a). \quad (2.7)$$

The last part of the assertion is a simple consequence.

For an upper bound on $h(a)$, $\mu(a)$ depending explicitly on a we use $x_a(t) \leq at^2/2$ to (2.3). Hence,

$$x_a(t) \geq a \int_0^t (t-s) \exp \left(- \int_0^s c \frac{a^p}{2^p} \tau^{2p} d\tau \right) ds = a \int_0^t (t-s) \exp \left(-c \frac{a^p}{(2p+1)2^p} s^{2p+1} \right) ds.$$

One can easily show that the function

$$\varphi(t) := \int_0^t (t-s) \exp(-ks^\alpha) ds, \quad k = \frac{ca^p}{2^p(2p+1)}, \quad \alpha = 2p+1$$

has a similar behaviour as x_a in the sense that its graph has an asymptote $x = h^*t - \mu^*$, where

$$h^* = \int_0^\infty \exp(-ks^\alpha) ds = \frac{\Gamma(1/\alpha)}{\alpha k^{1/\alpha}}, \quad \mu^* = \int_0^\infty s \exp(-ks^\alpha) ds = \frac{\Gamma(2/\alpha)}{\alpha k^{2/\alpha}} \quad (2.8)$$

and its graph sits above this line. Hence,

$$x_a(\tau) \geq a^{1-\frac{p}{2p+1}} \cdot \Gamma\left(\frac{1}{2p+1}\right) \left(\frac{2^p}{c \cdot (2p+1)^{2p}}\right)^{\frac{1}{2p+1}} \cdot \tau - a^{1-\frac{2p}{2p+1}} \cdot \Gamma\left(\frac{2}{2p+1}\right) (2p+1)^{\frac{1-2p}{2p+1}} \left(\frac{2^p}{c}\right)^{\frac{2}{2p+1}}.$$

$$x_a(\tau) \geq c_2 \cdot a^{\frac{p+1}{2p+1}} \cdot \tau - c_3 \cdot a^{\frac{1}{2p+1}}, \quad (2.9)$$

where

$$c_2 := \Gamma\left(\frac{1}{2p+1}\right) \left(\frac{2^p}{c \cdot (2p+1)^{2p}}\right)^{\frac{1}{2p+1}}, \quad c_3 := \Gamma\left(\frac{2}{2p+1}\right) (2p+1)^{\frac{1-2p}{2p+1}} \left(\frac{2^p}{c}\right)^{\frac{2}{2p+1}}. \quad (2.10)$$

Now, we are able to get appropriate estimates for $h(a)$.

Lemma 4. *For any $a > 0$,*

$$c_2 \cdot a^{(p+1)/(2p+1)} \leq h(a) \leq c_1 \cdot a^{(p+1)/(2p+1)}, \quad (2.11)$$

where

$$c_1 := \frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c_2(p+1))^{p/(p+1)}}.$$

Proof. For a lower bound we apply the estimate $x_a(\tau) \leq \frac{1}{2}a\tau^2$ to the equality

$$h(a) = a \int_0^\infty \exp \left(-c \int_0^s x_a(\tau)^p d\tau \right) ds. \quad (2.12)$$

This leads to the inequality

$$h(a) \geq a \int_0^\infty \exp \left(-\frac{ca^p}{2^p(2p+1)} s^{2p+1} \right) ds.$$

(2.8) for $k = \frac{ca^p}{2^p(2p+1)}$, $\alpha = 2p + 1$ gives the lower bound on h .

For an upper bound on h we use the lower estimate of x_a - (2.9) to the equality (2.12) and we get

$$\begin{aligned} h(a) &\leq a \int_0^{c_3/c_2 \cdot a^{-p/(2p+1)}} ds + \\ &a \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^\infty \exp \left(-c \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^s \left(c_2 a^{\frac{p+1}{2p+1}} \tau - c_3 a^{\frac{1}{2p+1}} \right)^p d\tau \right) ds \\ &\leq \frac{c_3}{c_2} a^{\frac{p+1}{2p+1}} + \frac{1}{c_2} a^{\frac{p}{2p+1}} \int_0^\infty \exp \left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt, \end{aligned}$$

where we used linear substitutions twice. At last, using (2.8) for $k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}}$, $\alpha = p + 1$ we have for any $a > 0$,

$$h(a) \leq \left(\frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c_2(p+1))^{p/(p+1)}} \right) \cdot a^{\frac{p+1}{2p+1}}.$$

The next lemma presents estimates for $\mu(a)$.

Lemma 5. *For any $a > 0$, constant $\mu(a)$ satisfies the following estimates:*

$$c_4 \cdot a^{1/(2p+1)} \leq \mu(a) \leq c_5 \cdot a^{1/(2p+1)},$$

where

$$\begin{aligned} c_4 &:= \frac{2^{2p/(2p+1)} \Gamma(2/(2p+1))}{(2p+1)^{(2p-1)/(2p+1)} c^{2/(2p+1)}}, \\ c_5 &:= \frac{1}{c_2^2} \left[\frac{c_3^2}{2} + \left(\frac{c_2}{c} \right)^{2/(p+1)} (p+1)^{(1-p)/(1+p)} \Gamma\left(\frac{2}{p+1}\right) + c_3 \left(\frac{c_2}{c \cdot (p+1)^p} \right)^{1/(p+1)} \Gamma\left(\frac{1}{p+1}\right) \right]. \end{aligned}$$

Proof. From (2.3) and (2.12) we get

$$\mu(a) = a \int_0^\infty s \exp \left(-c \int_0^s x_a(\tau)^p d\tau \right) ds. \quad (2.13)$$

Using the estimate $x_a(\tau) \leq \frac{1}{2}a\tau^2$ we get

$$\mu(a) \geq a \int_0^\infty s \cdot \exp \left(-\frac{ca^p}{2^p(2p+1)} s^{2p+1} \right) ds.$$

Using (2.8) for $k = \frac{ca^p}{2^p(2p+1)}$, $\alpha = 2p + 1$ we obtain the lower bound. On the other hand, from (2.9), (2.13) we have

$$\begin{aligned} \mu(a) &\leq a \int_0^{c_3/c_2 \cdot a^{-p/(2p+1)}} s ds + \\ &a \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^\infty s \exp \left(-c \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^s \left(c_2 a^{\frac{p+1}{2p+1}} \tau - c_3 a^{\frac{1}{2p+1}} \right)^p d\tau \right) ds \leq \\ &\leq \frac{c_3^2}{2c_2^2} a^{\frac{1}{2p+1}} + \frac{1}{c_2^2} a^{\frac{-1}{2p+1}} \int_0^\infty t \exp \left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt + \\ &+ \frac{c_3}{c_2^2} \int_0^\infty \exp \left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt, \end{aligned}$$

where we used linear substitutions twice. At last, using (2.8) for $k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}}$, $\alpha = p+1$ we have for any $a > 0$ we get the upper bound.

3 Main results

Now, we are able to prove the existence of a solution to (1.1).

Theorem 1. *The BVP (1.1) has a solution for any $\beta \geq 0$.*

Proof. The function $h : [0, \infty) \rightarrow \mathbb{R}$ is continuous on $(0, +\infty)$ by the continuous dependence of solutions of ODEs on initial conditions and locally uniform convergence of the integral

$$\int_0^\infty \exp\left(-c \int_0^s x_a(\tau)^p d\tau\right) ds.$$

By the estimates from lemma 4, we have

$$\lim_{a \rightarrow 0^+} h(a) = 0, \quad \lim_{a \rightarrow \infty} h(a) = +\infty.$$

Thus, for any $\beta > 0$, there exists $a > 0$ such that $h(a) = \beta$. For $\beta = 0$, it is obvious.

Finally, the uniqueness of the solution of (1.1) will be proved by using the ideas from [4]. For any $a > 0$ consider the one-to-one function $v_a : [0, h(a)^2) \rightarrow [0, \infty)$ such that $v_a(x'_a(t)^2) = x_a(t)$ for each $t \geq 0$. It is well defined since x_a and x'_a are increasing functions and it belongs to $C^2(0, h(a)^2)$. Substituting $y = x'(t)^2$, we shall find an ODE satisfied by v (we omit subscript a for simplicity).

$$x(t) = v(y), \quad x'(t) = v'(y)2x'(t)x''(t)$$

hence,

$$x''(t) = \frac{1}{2v'(y)}, \quad x'''(t) = -\frac{v''(y)}{2v'(y)^2} \cdot 2x'(t)x''(t) = -\frac{v''(y)\sqrt{y}}{2v'(y)^3}.$$

Put x and its derivatives in our ODE and find

$$v''(y) = \frac{cv(y)^p v'(y)^2}{\sqrt{y}}. \quad (3.1)$$

From boundary conditions on x we get

$$v(0) = 0, \quad v'(0) = \frac{1}{2a}, \quad \lim_{y \rightarrow h(a)^2-} v(y) = +\infty. \quad (3.2)$$

Now, we are in position to prove

Theorem 2. *The solution of (1.1) is unique.*

Proof. We need to show that the function h is one-to-one. Suppose that $h(a_1) = h(a_2)$, $a_2 > a_1$ and take v_1 and v_2 obtained by x_{a_1} and x_{a_2} , respectively, that is v_i satisfies (3.1) with boundary conditions (3.2) (for $a = a_i$, $i = 1, 2$). Put $w = v_1 - v_2$. Then $w(0) = 0$, $w'(0) = \frac{a_2 - a_1}{2a_1 a_2} > 0$. Notice that $w' > 0$ on the whole interval $(0, h(a_1)^2)$ – both function are defined on the same interval.

In fact, if it is not true, then there exists s in this interval such that $w' > 0$ on $(0, s)$ and $w'(s) = 0$. Hence $w(s) > w(0) = 0$ and

$$w''(s) = \lim_{\xi \rightarrow 0+} \frac{w'(s - \xi) - w'(s)}{-\xi} \leq 0.$$

On the other hand,

$$w''(s) = v_1''(s) - v_2''(s) = cs^{-1/2}(v_1(s)^p - v_2(s)^p)v'_{a_i}(s)^2$$

since $w'(s) = 0$ implies $v'_{a_1}(s) = v'_{a_2}(s)$. But $v_1(s)^p > v_2(s)^p$ from $w(s) > 0$ and this gives $w''(s) > 0$ – a contradiction. Thus, we have $w > 0$ and $w' > 0$ on $(0, h(a_1)^2)$.

Set $V_i = 1/v'_i$, $i = 1, 2$ and $W = V_1 - V_2$. We have, for any $y \in (0, h(a_1)^2)$,

$$W'(y) = V_1'(y) - V_2'(y) = -\frac{v_1''(y)}{v_1'(y)^2} + \frac{v_2''(y)}{v_2'(y)^2} = c \frac{v_2(y)^p - v_1(y)^p}{\sqrt{y}} < 0$$

from $w(y) > 0$. Hence,

$$W(y) < W(0) = \frac{1}{v'_1(0)} - \frac{1}{v'_2(0)} = 2(a_1 - a_2)$$

and

$$\lim_{y \rightarrow h(a_1)^2-} W(y) \leq 2(a_1 - a_2) < 0.$$

On the other hand, $V_i(x'_{a_i}(t)^2) = 2x''_{a_i}(t)$ implies

$$\lim_{y \rightarrow h(a_1)^2-} V_i(y) = \lim_{t \rightarrow \infty} V_i(x'_{a_i}(t)^2) = 2 \lim_{t \rightarrow \infty} x''_{a_i}(t) = 0$$

and, therefore,

$$\lim_{y \rightarrow h(a_1)^2-} W(y) = 0$$

which contradicts the previous inequality.

4 Numerical approach

All numerical methods cannot work on the infinite interval $[0, \infty)$ and we do not know the exact value of $a = x''(0)$ for the solution. Our earlier results make possible to find a finite interval $[0, T]$ for any positive value ϵ of the error control tolerance such that

$$x''(T) < \epsilon, \quad h(a) - x'(T) < \epsilon, \quad x(T) - (h(a)T - \mu(a)) < \epsilon.$$

Since all these functions decrease, all three inequalities hold for any $t > T$. First, by using estimates (2.11), we can find an interval $[a_{min}, a_{max}]$ such that $h(a_{min}) < \beta < h(a_{max})$. Next, by (2.1), we need

$$a_{max} \exp \left(-c \int_0^T x_{a_{min}}(\tau)^p d\tau \right) < \epsilon,$$

by (2.12), we should have

$$a_{max} \int_T^\infty \exp \left(-c \int_0^s x_{a_{min}}(\tau)^p d\tau \right) ds < \epsilon,$$

and by (2.3) and (2.13), we get

$$-a_{max} \int_T^\infty (T-s) \exp \left(-c \int_0^s x_{a_{min}}(\tau)^p d\tau \right) ds < \epsilon.$$

We do not know the function $x_{a_{min}}$ but we can use estimate (2.9) to get

$$a_{max} \exp \left(-c \int_0^T \left(c_2 \cdot a_{min}^{\frac{p+1}{2p+1}} \cdot \tau - c_3 \cdot a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \right) < \epsilon, \quad (4.1)$$

$$a_{max} \int_T^\infty \exp \left(-c \int_0^s \left(c_2 \cdot a_{min}^{\frac{p+1}{2p+1}} \cdot \tau - c_3 \cdot a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \right) ds < \epsilon, \quad (4.2)$$

$$-a_{max} \int_T^\infty (T-s) \exp \left(-c \int_0^s \left(c_2 \cdot a_{min}^{\frac{p+1}{2p+1}} \cdot \tau - c_3 \cdot a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \right) ds < \epsilon. \quad (4.3)$$

We start with a family of initial value problems

$$x_a(t) \text{ for } t \in [0, T], \quad x_a(0) = 0 = x'_a(0), \quad x''_a(0) = a, \quad a \in [a_{min}, a_{max}]. \quad (4.4)$$

If we approximate this solution in $[0, T]$ with an error less than ϵ , then the best approximation of x in $[T, \infty)$ is

$$x(t) = \beta t + (x_a(T) - \beta T).$$

The lower and upper bounds for the second derivative describe the shooting window - for each a the only solution in this direction exists at $t = T$, and the computed value of $x'(T)$ is more and more close to the expected limit value β . As long as β is contained between the computed values $x'(T)$ of the best two shots, we apply the classical bisection method:

If y and z are solutions such that $y''(0) < z''(0)$, and there is $y'(T) < \beta < z'(T)$, then the next problem to solve is (4.4) with $a = (y''(0) + z''(0))/2$.

Examples.

While solving the initial value problems we apply an adaptive Runge-Kutta-Fehlberg method RK45 [5], in which a tolerance parameter ϵ controls local error of the method. The values of ϵ range from 10^{-8} to 10^{-14} . For representing real values we use standard 16 – 17-digits double data type.

Numerical results for the classical Blasius equation $p = 1$, $c = 1/2$, and $\beta = 1$.

Here $a_{min} = 0.2694860459$, $a_{max} = 0.3420953216$. For $T = 14$ we get the all three inequalities (4.1), (4.2) and (4.3) for $\epsilon = 1.0e - 14$. Below N stands for a number of steps in RK45 (average):

ϵ	N	a	$ x''(0) - a $	$ 1 - x'(T) $	$x(T)$
1.0e-08	112	0.332057330068201	6.15e -09	1.2e-08	12.279212180321
1.0e-09	197	0.332057335646357	5.69e -10	1.1e-09	12.279212327474
1.0e-10	338	0.332057336149237	6.60e -11	1.3e-10	12.279212340740
1.0e-11	611	0.332057336210248	4.95e -12	9.9e-12	12.279212342350
1.0e-12	971	0.332057336214903	2.94e -13	5.7e-13	12.279212342472
1.0e-13	1831	0.332057336215154	4.19e-14	6.5e-14	12.279212342479
1.0e-14	3346	0.332057336215186	1.06e-14	5.5e-16	12.279212342480

The last value of a differs in two last digits from the one cited in [2]:

$$a = 0.33205733621519630.$$

The last two columns of the table have been computed for $\epsilon = 10^{-14}$. As there is $x''(T) = 7.68e - 13$, for $t > T$ the straight line approximation of the solution is the most effective.

Numerical results for the equation with $p = 7$, $c = 1/2$ and $\beta = 1$.

Here, $a_{min} = 0.3733978388$, $a_{max} = 0.3805482427$. As above for the tolerance $\epsilon = 1.0e - 14$, the interval $[0, 4]$ is sufficiently large and we get the following results by RK45 method:

ϵ	N	a	$ x''(0) - a $	$ 1 - x'(T) $	$x(T)$
1.0e-08	189	0.379398164451122	2.47e-08	3.5e-08	2.673055448977
1.0e-09	316	0.379398187063634	2.04e-09	2.9e-09	2.673055570853
1.0e-10	549	0.379398189005442	1.03e-10	1.4e-10	2.673055581319
1.0e-11	961	0.379398189086642	2.20e-11	3.1e-11	2.673055581757
1.0e-12	1688	0.379398189106905	1.69e-12	2.3e-12	2.673055581866
1.0e-13	2827	0.379398189108438	1.62e-13	1.9e-13	2.673055581874
1.0e-14	4634	0.379398189108571	2.91e-14	0.0e-14	2.673055581875

Remarks - as above. Here $x''(T) = 9.03e - 18$. The value of $a = 0.3793981891086 -$ here, all digits are true.

Numerical experiment for the equation with $p = 0.1$ $c = 1/2$, $\beta = 1$; taking $\epsilon = 10^{-14}$ we get $T = 50$. The proof of the existence and uniqueness result for $p < 1$ fails, since we cannot claim that the initial value problem (1.3) has a unique solution and that it depends continuously on a . Hence, function h can be multivalued. If one will prove the uniqueness, then, due to [7] p. 172, h will be continuous and all results of this paper will be true also for $p < 1$. The stability of numerical experiments cited below suggests it is the fact.

ϵ	N	a	$ x''(0) - a $	$ 1 - x'(T) $	$x(T)$
1.0e-08	142	0.443643205985427	2.16e-07	4.5e-07	48.05426086324
1.0e-09	256	0.443643403844908	1.78e-08	3.7e-08	48.05428058120
1.0e-10	466	0.443643420192529	1.49e-09	3.1e-09	48.05428221034
1.0e-11	839	0.443643421402885	2.80e-10	5.8e-10	48.05428233096
1.0e-12	1505	0.443643421660499	2.25e-11	4.7e-11	48.05428235664
1.0e-13	2669	0.443643421681506	1.49e-12	3.6e-12	48.05428235873
1.0e-14	4922	0.443643421683245	2.45e-13	2.2e-16	48.05428235890

Remarks as above. Here $x''(T) = 1.02e - 15$. The value of $a = 0.443643421683$ – here, all digits are true.

5 Conclusions

The authors know that our computation of the value of the second derivative of the solution are not more exact than others. However, the proposed method gives a possibility of controlling errors and it is very simple. We hope a similar approach can be applied for more general equations as $x''' + f(x) \cdot g(x'') = 0$ with qualitative assumptions on functions f and g .

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